Situations Conference 2009
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Draft of Situation I:

Prompt

In a general math class, having just discussed how to convert fractions to their decimal equivalents, students were asked to work on a few practice problems. The following conversation took place.

Student A: My group has question. … How do you go backwards?

Teacher: What do you mean by "go backwards?"

Student A: We know how to take $\frac{4}{5}$, do 4 divided by 5 and get 0.8. How do you start with 0.8 and get $\frac{4}{5}$?

Student B: Yeah, and what about $0.333\cdots = 0.\overline{3}$? How do you start with $0.333\cdots$ and get $\frac{1}{3}$? What fraction goes with $0.555\cdots$?

Commentary

The questions posed by this group of students address the equivalence of a fraction and its decimal counterpart. In other words, for fraction X to be equivalent to decimal number Y, fraction X implies decimal number Y, AND decimal number Y implies fraction X. The mathematical foci use place value, partitions of numbers, and properties of infinite geometric series to illustrate how one might address the latter of these two conditions.

Mathematical Foci

Mathematical Focus 1:

To convert a terminating decimal number to one of its fraction equivalents, we make use of place value.

Given A, a rational number where A is expressed in the form, $0.a_1a_2a_3\cdots a_n$, where $a_i \in \{0,1,2,3,4,5,6,7,8,9\}$ and n is a positive integer.

The place value of the right-most digit of A, a_n, is $10^n ths$ and serves as the denominator of a fraction equivalent of A.

$$A = 0.a_1a_2a_3\cdots a_n = \frac{a_1a_2a_3\cdots a_n}{10^n}$$

Mathematical Focus 2:

For infinitely repeating decimal numbers, a right-most digit does not exist and so place value is not helpful. To answer Student B's second question, what fraction goes with $0.555\cdots$?, one can make use of partitions.

$$\begin{aligned} \tfrac{2}{3} &= 0.666\cdots \\ &= (0.111\cdots)+(0.111\cdots)+(0.111\cdots)+(0.111\cdots)+(0.111\cdots)+(0.111\cdots) \\ &= 6(0.111\cdots) \end{aligned}$$

So,

$$\begin{aligned} 0.111\cdots &= \tfrac{1}{6}\, of\, \tfrac{2}{3} \\ &= \tfrac{1}{9} \end{aligned}$$

Since $0.111\cdots = \frac{1}{9}$, $0.555\cdots$ equals $\frac{5}{9}$.

Mathematical Focus 3:

To answer Student B's second question, "what fraction goes with $0.555\cdots$?," one can take advantage of the fact that $0.555\cdots$ is the sum of the geometric series, $\frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots$. Letting $N = 0.555\cdots = \frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots$, the following is true.

Since $N = \frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots$, $10N = 5+\frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots$. So by subtraction, $10N - N = 9N = 5$. Therefore, $N = \frac{5}{9}$.

Mathematical Focus 4:

To answer Student B's second question, "what fraction goes with $0.555\cdots$?," one can take advantage of the fact that $0.555\cdots$ is the sum of the geometric series, $\frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots$ and the formula for the sum of an infinite geometric series,

$S = \sum_{i=1}^{\infty} ar^i = \dfrac{ar}{1-r}$ where $0 < |r| < 1$.

Using sigma notation $0.555\cdots = \frac{5}{10}+\frac{5}{100}+\frac{5}{100}+\cdots = \sum_{i=1}^{\infty} 5\bullet\left(\frac{1}{10}\right)^i$. So, $a = 5$ and $r = \frac{1}{10}$ and therefore, the sum, $S = \dfrac{5\bullet\frac{1}{10}}{1-\frac{1}{10}} = \dfrac{\frac{5}{10}}{\frac{9}{10}} = \frac{5}{9}$.

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Draft of Situation II (derived from personal experience and some input from Glen Blume):

Prompt

A teacher wrote "$0.999\cdots$" and asked the class, "Besides itself, what does this number equal?" The ensuing discussion revealed four different positions.

(1) $0.999\cdots$ equals 1. It's just one of those facts I memorized from another class.
(2) $0.999\cdots$ equals 1. But I don't believe it.
(3) I'm convinced $0.999\cdots$ equals 1.
(4) $0.999\cdots$ can't equal 1, but it's very close to 1.

How might one "convince" students that $0.999\cdots$ equals 1?

Commentary

There are different methods one can use to show that $0.999\cdots$ equals 1. Mathematical Foci 1 to 3 reflect four methods. However, part of this situation is includes the notion of mathematical proof and how "convincing argument," at least in this situation, may differ from a "mathematical proof." I have had discussions with students in general math, pre-Algebra, Algebra 2, and Pre-Calculus about "$0.999\cdots = 1$" and have always had students leaving class saying something similar to position #2 above. Glen states that he had a similar experience with a class of secondary pre-service teachers.

Mathematical Foci

Mathematics Foci 1:

One way of showing $0.999\cdots$ equals 1 makes use of partitioning.

$$0.999\cdots = (0.333\cdots) + (0.666\cdots)$$

Since $0.333\cdots = \frac{1}{3}$ and $0.666\cdots = \frac{2}{3}$,

$$\begin{aligned} 0.999\cdots &= \tfrac{1}{3} + \tfrac{2}{3} \\ &= 1 \end{aligned}$$

Mathematics Foci 2:

The number, $N = 0.999\cdots$, is a special case of an infinite geometric series,

$S = ar + ar^2 + ar^3 + \cdots = \sum_{i=1}^{\infty} ar^i = \frac{ar}{1-r}$, where $0 < |r| < 1$.

$$N = 0.999\cdots = \tfrac{9}{10} + \tfrac{9}{100} + \tfrac{9}{1000} + \cdots = 9\bullet\tfrac{1}{10} + 9\bullet\left(\tfrac{1}{10}\right)^2 + 9\bullet\left(\tfrac{1}{10}\right)^3\cdots$$

So, $a = 9$ and $r = \frac{1}{10}$.

$$\text{Therefore, } S = \frac{9\bullet\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

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Mathematics Foci 3:

The number, $N = 0.999\cdots$, is a special case of an infinite geometric series, $S = ar + ar^2 + ar^3 + \cdots = \sum_{i=1}^{\infty} ar^i = \frac{ar}{1-r}$, where $0 < |r| < 1$. Rather than applying the formula directly, one apply a method by which $S = \frac{ar}{1-r}$ is derived to $N = 0.999\cdots$.

$$\begin{array}{rl} N & = 0.999\cdots \\ -10N & = 9.999\cdots \\ \hline -9N & = -9 \\ N & = \frac{-9}{-9} = 1 \end{array}$$